Mathematics for Engineers I. seminar

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Linear algebra

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1.) Let

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ -1 \\ 7 \end{pmatrix},$$

$$\mathbf{u} = \begin{pmatrix} -1+i \\ 2i \end{pmatrix}, \mathbf{v} = \begin{pmatrix} \sqrt{2}+i \\ i^2-3i \end{pmatrix}, \mathbf{w} = \begin{pmatrix} i-1 \\ i-2 \\ 1 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} 0 \\ -1 \\ 7 \end{pmatrix}.$$

Do the following calculations!

$$\mathbf{a}+\mathbf{b}, \quad -5\mathbf{c}, \quad 12\mathbf{a}+4\mathbf{b}, \quad 3\mathbf{c}+\mathbf{d}, \quad \|\mathbf{b}\|, \quad \|\mathbf{c}+\mathbf{d}\|, \quad \mathbf{u}-i\mathbf{v},$$

$$(3+2i)\mathbf{w}-i\mathbf{z}, \quad \mathbf{v}-3i\mathbf{u}, \quad \|\mathbf{w}\|, \quad \|\mathbf{z}\|, \quad \|i\mathbf{z}+w\|.$$

2.)

- Determine the angle of the vectors **a** and **b**!
  (a) $\mathbf{a}^T = (2, 1)$, $\mathbf{b}^T = (1, 3)$
  (b) $\mathbf{a}^T = (3, \sqrt{3})$, $\mathbf{b}^T = (2, 0)$
  (c) $\mathbf{a}^T = (1, 3, 1)$, $\mathbf{b}^T = (-4, -2, 0)$
  (d) $\mathbf{a}^T = (-2, 3, 4)$, $\mathbf{b}^T = (-6, -4, 2)$

- Give the number λ such that **a** will be perpendicular to **b**!

3.)
  (a) $\mathbf{a}^T = (1, 1)$, $\mathbf{b}^T = (-2, \lambda)$
  (b) $\mathbf{a}^T = (4, 2, 1)$, $\mathbf{b}^T = (-4, -2, \lambda)$
  (c) $\mathbf{a}^T = (1, 2, 1, 1)$, $\mathbf{b}^T = (-4, -2, 2, \lambda)$
  (d) $\mathbf{a}^T = (1, \lambda, \lambda)$, $\mathbf{b}^T = (-3, -2, \lambda)$

4.) Prove that the vectors $[1, 2]$, and $[-3, 2]$ are linearly independent in $\mathbb{R}^2$!

5.) Which of the following set of vectors are linearly dependent or independent in $\mathbb{R}^3$?

- $\mathbf{v}_1^T = [1, 0, 1]$, $\mathbf{v}_2^T = [1, 1, 1]$;
- $\mathbf{v}_1^T = [1, 0, 0]$, $\mathbf{v}_2^T = [0, 1, 0]$, $\mathbf{v}_3^T = [1, 1, 1]$;
- $\mathbf{v}_1^T = [1, 2, 2]$, $\mathbf{v}_2^T = [-1, -1, 3]$, $\mathbf{v}_3 = [2, 3, 0]$;
- $\mathbf{v}_1^T = [1, 1, 2]$, $\mathbf{v}_2^T = [2, 3, -1]$, $\mathbf{v}_3 = [-1, 2, -17]$;
- $\mathbf{v}_1^T = [1, 1, 2]$, $\mathbf{v}_2^T = [-2, -2, -4]$;
- $\mathbf{v}_1^T = [-1, 2, 1]$, $\mathbf{v}_2^T = [2, -3, 1]$, $\mathbf{v}_3 = [1, 1, 2]$.

6.) Determine the subspaces in $\mathbb{R}^2$ generated by the following vectors!

- $[1, 0]$, $[0, 1]$;
- $[-2, 0]$, $[0, 1]$;
- $[1, -2]$, $[2, 4]$;
- $[3, 2]$, $[]$;
- $[1, 2]$, $[3, 2]$;
- $[1, 0]$, $[2, 4]$, $[0, 2]$.

7.) Determine the subspaces in $\mathbb{R}^3$ generated by the following vectors!

- $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$;
- $[1, 0, 0]$, $[1, 1, 1]$;
- $[1, 1, 0]$, $[1, 0, 1]$;
- $[2, 2, 1]$, $[3, 1, 3]$, $[-1, 2, 0]$;
- $[0, 0, 1]$;
- $[2, 2, 0]$, $[1, 1, 1]$, $[1, 1, -1]$.

8.) Let

$$\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix},$$

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \pi \\ i-1 \\ 3-2i \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1+i \\ -1-i \\ \lambda \end{pmatrix},$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -1 \\ 2 & -6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix},$$

Execute the following operations:

$B\mathbf{a}$, $B\mathbf{b}$, $A\mathbf{x}$, $\mathbf{x}^T B$, $\mathbf{v}^T\mathbf{d}$, $C\mathbf{w}$, $\mathbf{w}^T C\mathbf{d}\mathbf{a}^T$, $\mathbf{b}^T B$, $\mathbf{a}^T\mathbf{b}$, $\mathbf{a}\mathbf{b}^T$, $\mathbf{w}^T\mathbf{w}$, $\mathbf{w}\mathbf{w}^T$!

9.) Plot the following expressions on the plane

(a) x és Ax,

(b) x és Bx

(c) x és Cx

(d) x és ACx

(e) x és BCx

if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}!$$

10.) Let's consider the following matrices:

$$A = \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 \\ -2 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -3 \\ -3 & -5 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 4 & 2 & 1 \\ -2 & -3 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & -2 & 5 \\ -1 & -1 & 0 \\ -2 & 1 & 1 \end{pmatrix}!$$

Find the value of the following expressions!

$$A+B, \quad -3B, \quad 2A+3C, \quad A^T, \quad B^T, \quad C^T, \quad (A+B)^T, \quad (3B)^T,$$

$$AB, \quad BA, \quad AE, \quad EA, \quad AC, \quad A(B+C), \quad C(2A-B), \quad A^TA,$$

$$FG, \quad 2F-3G, \quad F^T-G, \quad G+2F^T, \quad G^2-2FG+F^2, \quad (G-F)^2$$

11.) The quarterly sales of Jute, Cotton and Yarn for the year 2002 and 2003 are given below.

$$\begin{pmatrix} 20 & 25 & 22 & 20 \\ 10 & 20 & 18 & 10 \\ 15 & 20 & 15 & 15 \end{pmatrix}, \quad \begin{pmatrix} 10 & 15 & 20 & 20 \\ 5 & 20 & 18 & 10 \\ 8 & 30 & 15 & 10 \end{pmatrix}.$$

Find the total quarterly sales of Jute, Cotton and Yarn for the two years.

12.) Determine the determinants and the inverse of the following matrices!

$$\begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}, \quad \begin{pmatrix} 5 & -2 \\ 2 & 8 \end{pmatrix}, \quad \begin{pmatrix} -4 & 1 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix}, \quad \begin{pmatrix} -2 & 4 \\ -1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix},$$

$$\begin{pmatrix} -3 & 1 \\ 5 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} -1 & 7 \\ 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 5 \\ -1 & 7 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -5 \\ -3 & 10 \end{pmatrix}, \quad \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

13.) Determine the determinants and the inverse of the following matrices!

$$\begin{pmatrix} 4 & 2 & 1 \\ -2 & -3 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}, \quad \begin{pmatrix} -2 & 3 & 19 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ 2 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 & 0 & -1 \\ 1 & 0 & 5 & -1 \\ 1 & -2 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

14.) Solve the linear systems of equations below!

(a)

$$\begin{aligned} x_1 - 2x_2 &= 6 \\ 3x_1 - x_2 &= 13 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - 3x_2 &= 1 \\ 4x_1 - x_2 &= 7 \end{aligned}$$

(c)

$$\begin{aligned} -x_1 - 5x_2 &= 7 \\ 4x_1 - 3x_2 &= -5 \end{aligned}$$

(d)

$$\begin{aligned} -3x_1 + 2x_2 &= -4 \\ 6x_1 - 4x_2 &= 8 \end{aligned}$$

(e)

$$\begin{aligned} 2x_1 - x_2 &= -4 \\ 6x_1 - 3x_2 &= 9 \end{aligned}$$

(f)

$$\begin{aligned} -2x_1 - 5x_2 &= 3 \\ x_1 - 3x_2 &= 4 \end{aligned}$$

10.) Write the previous systems in the form $Ax = b$!

15.) Solve the linear systems below!

(a)

$$\begin{aligned} 2x_1 + 2x_2 \quad -3x_3 &= 0 \\ x_1 + 5x_2 \quad +2x_3 &= 1 \\ -4x_1 \qquad\quad +6x_3 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 + 2x_2 \quad -3x_3 &= 0 \\ x_1 + 5x_2 \quad +2x_3 &= 0 \\ -4x_1 \qquad\quad +6x_3 &= 0 \end{aligned}$$

(c)

$$\begin{aligned} 2x_1 + 2x_2 \quad -3x_3 &= 0 \\ x_1 + 5x_2 \quad +2x_3 &= 1 \end{aligned}$$

(d)

$$2x_1 + 2x_2 \quad -3x_3 = 0$$

16.) Solve the linear systems below!

(e)

$$\begin{aligned} x_1 \qquad\quad -x_3 &= 0 \\ -2x_1 + 3x_2 - x_3 &= 0 \\ -6x_1 + 6x_2 \qquad &= 0 \end{aligned}$$

(f)

$$\begin{aligned} x_1 \qquad\quad -x_3 &= 1 \\ -2x_1 + 3x_2 - x_3 &= 0 \\ -6x_1 + 6x_2 \qquad &= -2 \end{aligned}$$

(g)

$$\begin{aligned} 3x_1 - 6_2 - x_3 - x_4 &= 7 \\ -x_1 + 2x_2 + 2x_3 + 3x_4 &= 1 \\ 4x_1 - 8x_2 - 3x_3 - 2x_4 &= 6 \end{aligned}$$

(h)

$$\begin{aligned} 3x_1 - 6_2 - x_3 - x_4 &= 5 \\ -x_1 + 2x_2 + 2x_3 + 3x_4 &= 3 \\ 6x_1 - 8x_2 - 3x_3 - 2x_4 &= 1 \end{aligned}$$

17.) Determine the eigenvalues and eigenvectors of the following matirces!

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}, \quad \begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} -6 & 2 & -2 \\ 15 & 5 & 7 \\ 21 & 3 & 9 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}, \quad \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}.$$

18.) Determine the eigenvalues and eigenvectors of the following matirces!

$$\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}, \quad \begin{pmatrix} 3 & -1 & 0 \\ 6 & -3 & 2 \\ 8 & -6 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & -2 \\ -4 & 4 & 4 \\ -2 & 1 & 0 \end{pmatrix}.$$

Vectors in Matlab

It makes a distinction between row- and column vectors.

Creating row vectors

The most direct way is to put the values that you want in the vector in square brackets, separated by either spaces or commas. For example, both of the assignment statements below create the same vector *v*:

```
v = [-1.2,1,2,3.4]
v = [-1.2 1 2 3.4]
```

Numbering of the coordinates starts with 1 and `a(i)` denotes the *i*th coordinate of the vector *a*.

`length(a)` denotes the number of the coordinates of the vector *a*

`a = []` denotes the empty vector

Vectors as regular sequences

With the colon operator

- the vector $b = (1, 2, 3, 4, 5)$:

```
b = 1:5
```

- the vector $c = (5, 4, 3, 2, 1)$

```
c = 5:-1:1
```

- the vector $d = (2, 2.2, 2.4, 2.6, 2.8, 3)$

```
d = 2:0.2:3
```

In general:

`x = first element: step size: last element`,

where the step size can be negative too, or

`x = first element: last element`

in this case the default step size is 1.

Vectors as regular sequences

With the `linspace`function

- the vector $e = (1, 1.2, 1.4, 1.6, 1.8, 2)$

```
e=linspace(2,2,6)
```

- the vector f which has 100 coordinates

```
f=linspace(1,2)
```

In general

```
x = linspace(first element, last element, number of elements),
```

where the coordinates follow each other with the same step size

```
x = linspace(first element, last element)
```

the number of coordinates is 100 in the latter

Column vectors

Defining a column vector

- counting its elements (use semicolons between the coordinates)

```
1  m = [-3;0;7]
```

- transposing a row vector

```
1  n = [1 -2 4 -1]'
```

  (the sign ' results conjugate transpose, the transpose of a vector without conjugation is `transpose (a)` or `a.'`)

`x(i)` and the `length(x)` gives the ith coordinate of x and the length (number of coordinates) of x respectively

`size(x)` gives the size of x which is `[1 length(x)]` for row vectors and `[length(x) 1]` for column vectors

Constructing vectors from other vectors

- concatenation of two row vectors: `[a,b]`
- concatenation of two column vectors: `[m;n]`
- extension of a row vector with new elements: `[-4 a 3 -1]`
- extension of a column vector with new elements: `[1;m;-3]`
- the vector which contains the first, the fourth and the fifth coordinates of the vector *h*: `h([1 4 5])`
- omitting the second coordinate of the vector *h*: `h(2)=[]`
- omitting the second and the fourth coordinates of the vector *h*: `h([2 4])=[]`

Important remark: If `a=[-1 3 2]`, then the result of the command `a(6)=4` will be the vector `a=[-1 3 2 0 0 4]`.

Some useful functions

- `min(x)` and `max(x)` are the smallest and the greatest element of the vector x respectively
- `sort(x)` rearrange the coordinates of x in increasing order
- `sort(x,'descend')` rearrange the coordinates of x in decreasing order
- `flip(x)` coordinates of x in reverse order
- `length(x)` the number of the coordinates of x
- `sum(x)` the sum of the coordinates of x
- `prod(x)` the product of the coordinates of x
- `mean(x)` the average of the coordinates of x
- `x(i)` the ith coordinate of x
- `x(1:3)` the first three coordinates of x
- `x(3:end)` the coordinates of x from the third till the last

Operations with vectors

If a and b have the same type and size, then

- `a+b` and `a-b` are their sum and difference, respectively
- `x=a+1` is the same type and the same size as a and the ith coordinate of x is $x_i = a_i + 1$
- `x=a.^2` is the same type and the same size as a and the ith coordinate of x is $x_i = a_i^2$
- `x=a.*b` is the same type and the same size as a and b moreover, the ith coordinate of x is $x_i = a_i b_i$
- `x=a./b` is the same type and the same size as a and b moreover, the ith coordinate of x is $x_i = \frac{a_i}{b_i}$
- `dot(a,b)` is the inner product of the vectors a and b

Important remark: The dot before the operation means componentwise execution.

`sin`, `cos`, `tan`, `exp`, `log`, `sqrt`, `abs` etc. are executed always componentwise.

Exercises

- Generate the following vectors in the simplest way:
  1. $a = (1, 2, \ldots, 30)$
  2. $b = (2, 4, 6, \ldots, 100)$
  3. $c = (2, 1.9, 1.8, \ldots, 0)$
  4. $d = (0, 3, 6, \ldots, 27, 30, -100, 30, 27, \ldots, 6, 3, 0)$
  5. $e = (\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{20})$
  6. $f = (\frac{1}{2}, \frac{2}{3}, \ldots, \frac{19}{20})$
- Let $x = [1 : 100]$. Create y from x such that
  1. y has the coordinates of x in the reverse order,
  2. y has the first five coordinates of x,
  3. y has the same coordinates of x except for the fourth coordinate,
  4. y has the same coordinates of x except for the fourth, the seventy-second and the ninety-third coordinates,
  5. y contains such coordinates of x which has odd line number,
  6. y has the coordinates of x with line number two, five, seventeen and eighty-one.

Exercises

Let x be a row vector. Without the `for` command construct the vector y with ith coordinate is

1. $x(i) + 2$,
2. $x(i)^2$,
3. $1/x(i)$,
4. $\sin(x(i))^3 - 1)$,
5. $x(i) - i$.

Constructing matrices in Matlab

- Enumerating its elements

```
A = [1,2,3;4,5,6;7,8,9]
A = [1 2 3; 4 5 6; 7 8 9]
```

Elements in the same row separated by commas or spaces, rows separated by semicolons.

`A(i,j)` denotes the (i,j)th element of the matrix A.

- Concatenating vectors:

```
a = [1 -2 0];
b = [2 3 -2];
A = [a;b]
```

- Extension of matrices by vectors:

```
B = [A;[1 -2 2]]; C = [A, [1; -2]];
```

- Concatenating matrices: define at first two matrices G, H and concatenate them executing the commands:

```
G = [1 2; 0 1; 1 1]; H = [4 4; 5 5; 6 6];
A = [G H]
B = [G;H]
G(3,4)=2
```

The last matrix will be bigger, than G! Be careful because Matlab does not send

Element, rows, columns, submatrices

- `size(A)` is the size of A (two dimensional vector)
- `length(A)` is a number which is max{row numbers, column numbers}
- `A(i,j)` is the i, jth element of A
- `A(i,:)` is the ith row of A
- `A(:,j)` is the jth column of A
- `A(2:3,:)` the second and the third row of A
- `A(2:3,[1 3])` is a matrix which is the intersection of the second and third row with the first and the third column of A

Modification of a matrix

Cancelling rows and columns

- `A(i,:)=[]` omitting the *i*th row
- `A(:,j)=[]` omitting the *j*th column
- `A([1 3],:)=[]` dropping out the first and the third rows
- `A(:,[1 3])=[]` dropping out the first and the third columns

Interchanging rows and columns

- Interchanging the *i*th and the *j*th rows `A([i,j],:)=A([j,i],:)`
- Interchanging the *i*th and the *j*th columns
  `A(:,[i,j])=A(:,[j,i])`

Transformation a matrix into a vector `A(:)` is a column vector which contains the columns of A

Some built-in matrices in Matlab

- `eye(n)` the $n \times n$ unit matrix
- `eye(n,m)` an $n \times m$ matrix which contains the $\min\{m, n\} \times \min\{m, n\}$ unit matrix
- `ones(n,m)` the full 1 $n \times m$ matrix
- `zeros(n,m)` the full 0 $n \times m$ matrix

Some useful functions for matrices

- `numel(A)` the number of the elements of A
- `size(A)` the size of A
- `length(A)` is max{number of rows, number of columns}

Operations with matrices and vectors

Let A and B matrices (compatible size) and c be a scalar, then

```
A+B,  A-B,  c*A,  A*B,  A^2
```

are the usual matrix operations.

The result of `A+c` will be a matrix, same size as A and c is added to every elements of A.

The result of

`A/B` and `A\B`

will be AB^{-1} and $A^{-1}B$.

Operations with matrices and vectors

Elementwise operations

The elementwise operation is denoted by a dot . before the usual arithmetic operation, e.g.:

- the ijth element of `A.*B` is $a_{ij} b_{ij}$,
- the ijth element of `A.^2` is a_{ij}^2,
- the ijth element of `A./B` is a_{ij}/b_{ij}.

The built in Matlab functions, such as `log`, `sin`, `abs` etc can be called with matrix argument.

Exercises

Let `x = [-1 4 0]`, `y = [3 -2 5]` and `A = [-3 1 -4;6 2 -5]`.

Which can be executed from the following operations? Give the result if the task is executable and give explanation if it is not executable.

1. `z = [x,y]`
2. `z = [x;y]`
3. `z = [x',y']`
4. `z = [x';y']`
5. `z = [A,x]`
6. `z = [A;x]`
7. `z = [x;A;y]`
8. `z = [A';x]`
9. `z = [A',x]`
10. `z = [A',x']`
11. `x+y`
12. `x+y'`
13. `A+y`
14. `A+2`
15. `x/y`
16. `x./y`
17. `A^2`
18. `A.^2`

Exercises

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Construct from A the following matrix B:

1. omit the first row of A,
2. omit the second and the fourth columns of A,
3. omit the last row and the last column of A,
4. A is written twice beside itself,
5. transpose of A,
6. the second column is interchanged with the fourth column of A,
7. the squares of the elements of A.

Exercises

1. increase all the elements of A with three,
2. take the square all the elements of A,
3. take the sin all the elements of A,
4. change the second element of the first row with -2,
5. change the second row of A with `[-1 0 -2 3]`.

Exercises

1. Construct the following matrix using a short Matlab command!

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 \\ 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 \end{bmatrix}$$

2. Examine the result of the following commands using the previous matrix A!

- `sum(A)`
- `sum(A,2)`
- `reshape(A,6,4)`
- `max(A)`
- `max(A,[],2)`
- `max(A,2)`
- `flipud(A)`
- `fliplr(A)`
- `size(A)`
- `length(A)`

Linear algebra with Matlab

Using Matlab determine the linear dependency or independency of the following system:

$$a = \begin{bmatrix} 16 \\ 5 \\ 9 \\ 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 11 \\ 7 \\ 14 \end{bmatrix}, \qquad c = \begin{bmatrix} 3 \\ 10 \\ 6 \\ 15 \end{bmatrix}$$

First solution: let's construct the matrix below, and calculate its rank with the command `rank(A)`.

$$A = \begin{bmatrix} 16 & 2 & 3 \\ 5 & 11 & 10 \\ 9 & 7 & 6 \\ 4 & 14 & 15 \end{bmatrix}$$

Second solution: Call the `rref` function with the previous matrix A. The result will be the form of A after Gauss-Jordan elimination.

Exercise

Adjoin the transpose of the vector `d=[13 8 12 1]` to the previous system. Examine the linear dependency and independency of the new system. Use also the `det` function.

Exercise

- Find two linearly independent vectors in the following system, and combine linearly the rest from these.

$$v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Exercise

- Find three linearly independent vectors in the following system, and combine linearly the rest from these.

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 7 \\ -2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

Example

Solve the linear system of equations $Ax = b$ with Matlab if

$$A = \begin{bmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ -12 \end{bmatrix}$$

Solution: use the `backslash` operator

```
A = [-2 -1 4;2 3 -1;-4 -10 -5]
b = [3;1;-12]
x = A\b
```

Be careful, give b as a column vector!

We can solve this system applying the `rref` function to the augmented matrix `[A,b]`.

Example

Solve the linear system of equations $Ax = b$ with Matlab if

$$A = \begin{bmatrix} -4 & -4 & 2 \\ -2 & -7 & 3 \\ 2 & 12 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} -2 & 6 & -13 \end{bmatrix}$$

Solution: Using the `backslash` operator, we get a warning message:

`Warning: Matrix is singular to working precision`

The determinant of the system is zero, so we cannot solve the problem in this way.

With the `rref` function we really get that the columns of A constitute a linearly dependent system, however, b is in the linear hull of the columns. In this case it has infinitely many solutions e.g.

$$\begin{bmatrix} 1.9 \\ -1.4 \\ 0 \end{bmatrix}$$

We can determine all solutions with the `null` function. This give a basis in the nullspace (kernel) of A.

Exercises

1. $$A = \begin{bmatrix} 2 & -3 & 1 & 1 \\ -1 & 3 & 4 & 7 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

2. $$A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 5 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} -5 \\ 24 \\ -23 \end{bmatrix}$$

3. $$A = \begin{bmatrix} 2 & 1 & 5 & 0 \\ -3 & 4 & -13 & 22 \\ 5 & -1 & 16 & -16 \\ 1 & 1 & 2 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 12 \\ 81 \\ -33 \\ 15 \end{bmatrix}$$

The `rats(x)` or the `format rat` commands results fractional form numbers.

Example

Solve the linear system of equations $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 12 \end{bmatrix}$$

Solution: using the `backslash` operator we have `x^T=[1 2.7]`, which is not a solution of the system. This is caused by the fact that this system is overdetermined (more equations, than unknown variables). In this case the `backslash` operator gives back the coefficients of the closest vector to b in the subspace generated by the column of A.
With the `rref([A b])` command we have that the rank A is two and the rank of the augmented matrix is three, so there is no solution of this system.

More than one vector on the right-hand side

Example

Solve the linear systems of equations $Ax = b$ and $Ax = c$ if

$$A = \begin{bmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ -12 \end{bmatrix}, \quad c = \begin{bmatrix} 17 \\ 1 \\ -42 \end{bmatrix}$$

More than one vector on the right-hand side

Solution: The two systems has the same matrix, so we can solve them simultaneously using the `x = A\[b c]` command.

```
A = rand(10000);
b = ones(10000,1);
c = zeros(10000,1);
tic;x = A\[b c];toc %tic and toc functions work
    together to measure elapsed time
tic;x = A\b; xx = A\c; toc
```

```
Elapsed time is 9.447803 seconds.
Elapsed time is 18.108534 seconds.
```

7.69 GB memory and Intel(R) Core(TM) i5-3470 CPU was used.

Inverse of matrices

The `inv(A)` command calculates the inverse of the matrix A, if it is a square matrix. If A is not a square matrix or its determinant is close to zero, then Matlab sends an error message or a warning message. Calculation of the inverse of large matrices can be costly.

Eigenvalues, eigenvectors

Determine the eigenvalues and eigenvectors of the following matrices!

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}, \quad \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -5 \\ 1 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} -6 & 2 & -2 \\ 15 & 5 & 7 \\ 21 & 3 & 9 \end{bmatrix}, \quad \begin{bmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{bmatrix}, \quad \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$$

Eigenvalues, eigenvectors

$$\begin{bmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 0 \\ 6 & -3 & 2 \\ 8 & -6 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 \\ -4 & 4 & 4 \\ -2 & 1 & 0 \end{bmatrix}$$

Matlab example

Determine the eigenvectors and eigenvalues of the following matrix with Matlab!

$$A = \begin{bmatrix} -6 & 2 & -2 \\ 15 & 5 & 7 \\ 21 & 3 & 9 \end{bmatrix}$$

Eigenvalues and eigenvectors in Matlab

Solution: Use the `eig` function!

```
A = [-6 2 -2;15 5 7; 21 3 9]
u = eig(A)
[V U] = eig(A)
```

The second row gives only the eigenvalues. The third row results two matrices, the first contains the eigenvectors of A (column vectors), the second is a diagonal matrix and it contains the eigenvalues.

Exercise

$$A = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -1 & 4 & 2 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{bmatrix}$$